Optimal guidance and estimation of a 1D diffusion process by a team of mobile sensors

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Motivation

- Autonomous vehicles: monitoring and control of large-scale spatiotemporal processes.
- Modelling of the process: partial differential equations
  - e.g., advection-diffusion

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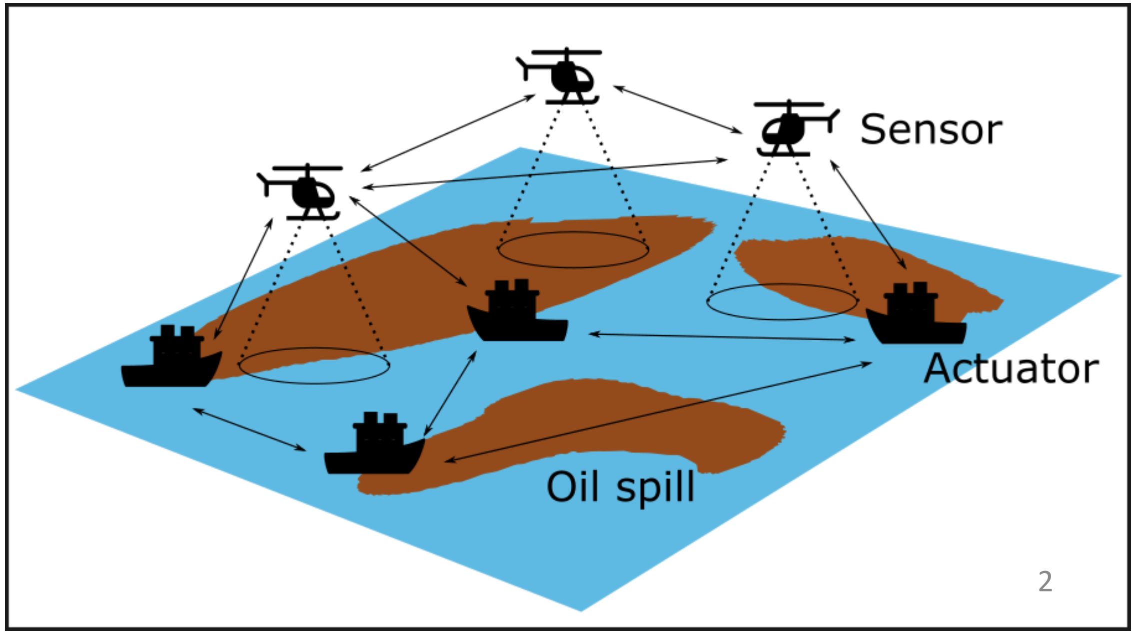

Relation to prior work

Sensor type		Representative work	Approach
Stationary	Boundary	[Smyshlyaev and Krstic, 2005]	Backstepping
		[Wang et al., 2017]	Luenberger
		[Moura and Fathy, 2017]	LQE
	In-domain	[Bensoussan, 1972]	Minimum trace
		[Demetriou and Borggaard, 2004]	Enhanced observability
		[Demetriou, 2017]	Centroidal Voronoi tessellation
		[Veldman et al., 2020]	Geometric rules
Mobile		[Demetriou and Hussein, 2009]	Lyapunov-based methods
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Our contributions to date:

1. Formulate an optimization problem: uncertainty + mobility penalty
2. Establish the conditions of the existence of a solution to the proposed problem
3. Analyze the sensor noise and mobility penalty's impact on the performance of the proposed guidance

		[Carotenuto et al., 1987]	Optimization
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Sensor dynamics and diffusion process

- Assume linear and first-order dynamics of the mobile sensors

$$\dot{\zeta}_i(t) = a_i \underset{\substack{\uparrow \\ \text{position}}}{\zeta_i(t)} + b_i \underset{\substack{\uparrow \\ \text{guidance}}}{p_i(t)}$$

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- Consider a 1D diffusion equation with unknown initial condition and known boundary condition:

$$\frac{\partial z(x,t)}{\partial t} = a\frac{\partial^2 z(x,t)}{\partial x^2} + D(x,t)\underbrace{w(t)}_{\text{Gaussian white noise}}$$

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$$y(t) = \int_\Omega \mathbb{B}_{\zeta(t),r}(x)z(x,t)\mathrm{d}x + \underbrace{v(t)}_{\text{Gaussian white noise}}$$

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position guidance

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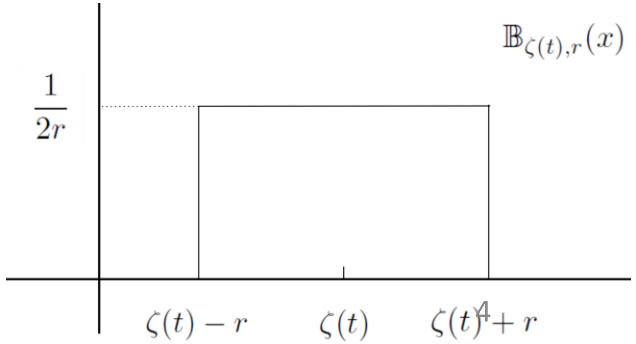

Infinite-dimensional system: Kalman filter

- Abstract linear system representation:

$$\begin{cases} \dot{\mathcal{Z}}(t) = \mathcal{A}\mathcal{Z}(t) + \mathcal{D}(t)w(t) \\ y(t) = \mathcal{C}^{\star}_{\zeta(t)}\mathcal{Z}(t) + v(t) \end{cases}$$

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- Kalman filter[1]

covariance of
measurement noise

$$\dot{\hat{\mathcal{Z}}}(t) = \mathcal{A}\hat{\mathcal{Z}}(t) + \mathcal{P}(t)\mathcal{C}_{\zeta(t)}R^{-1}(y(t) - \hat{y}(t))$$

$$\hat{\mathcal{Z}}(t_0) = \hat{\mathcal{Z}}_0,$$

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with operator-valued differential Riccati equation:

$$\dot{\mathcal{P}}(t) = \mathcal{A}\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A}^\star + \mathcal{D}(t)Q\mathcal{D}^\star(t) - \mathcal{P}(t)\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}^\star_\zeta(t)\mathcal{P}(t)$$

incremental covariance
of state noise

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Problem formulation

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- Uncertainty evaluated by the trace of the covariance operator[1]

$$\mathrm{Tr}(\mathcal{P}(t)) = \mathbb{E}\left[\|\mathcal{Z}(t) - \hat{\mathcal{Z}}(t)\|^2_{\mathcal{H}}\right]$$

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- Existence of a solution is established when the kernel of the observation function $\mathbb{B}_{\zeta(t),r}(x)$ is continuous w.r.t. sensor location[2].

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[2] J. A. Burns and C. N. Rautenberg, "The infinite-dimensional optimal filtering problem with mobile and stationary sensor networks," Numerical Functional Analysis and Optimization, vol. 36, no. 2, pp. 181–224, 2015.

Pontryagin’s maximum principle

Consider the Hamiltonian:

$$H(t) = \mathrm{Tr}(\mathcal{P}(t)) + \frac{1}{2}p^T(t)\gamma p(t) + \lambda^T(t)(a\zeta(t) + bp(t))$$

Optimality conditions:

$$\dot{\zeta}^*(t) = a\zeta^*(t) + bp^*(t),\ \ \zeta^*(0) = \zeta_0,$$
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Derivative of $\mathrm{Tr}(\mathcal{P}(t))$

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- $\Lambda(t)$: Fréchet derivative of $\mathcal{P}(t)$ with respect to $\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^*(t)$ [1]

$$\Lambda h(t) = -\int_0^t S(t-s)\big((\Lambda h)\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^\star\mathcal{P} + \mathcal{P}\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^\star(\Lambda h) + \mathcal{P}h\mathcal{P}\big)(s)S^\star(t-s)\mathrm{d}s,$$
$$\Lambda(0) = 0,$$

where $h(t)$ is a trace-class operator.

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- Apply chain rule

$$\frac{\partial \mathrm{Tr}(\mathcal{P}(t))}{\partial \zeta_i(t)} = \mathrm{Tr}(\Lambda(t) \circ D_{\zeta_i(t)}\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^\star(t)),$$

$D_{\zeta_i(t)}\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^*(t)$: Fréchet derivative of the operator $\bar{\mathcal{C}}_\zeta\bar{\mathcal{C}}_\zeta^*(t)$ with respect to the location of sensor i.

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- Galerkin approximation with orthonormal sinusoidal basis functions

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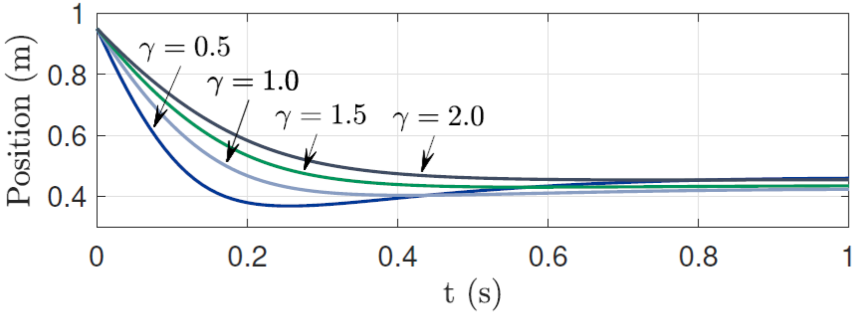
Simulation: single sensor

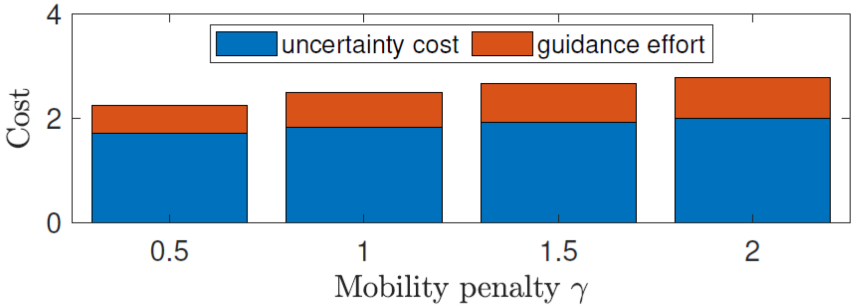
- Two import parameters in the problem formulation:
  - Sensor noise covariance R: smaller value means better performance
  - Mobility penalty γ: smaller value yields swifter vehicle
- Two cost components
  - Uncertainty cost: $\int_0^{t_f} \mathrm{Tr}(\mathcal{P}(t))\mathrm{d}t$
  - Guidance effort: $\frac{1}{2}\int_0^{t_f} p^T(t)\gamma p(t)\mathrm{d}t$
- Guidance in comparison in Monte Carlo simulation:

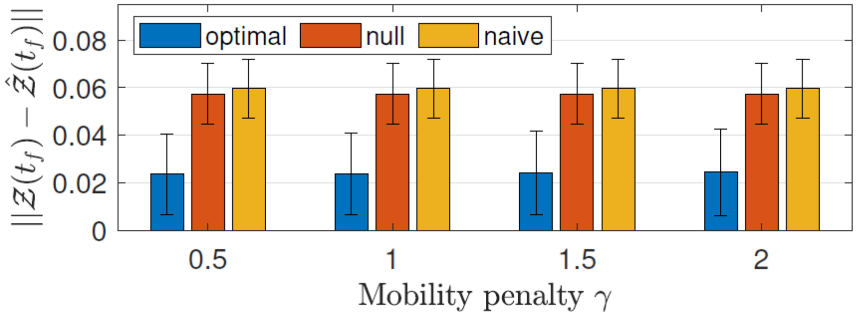
Guidance	Trajectory
Optimal	Optimal trajectory steered by numerically computed optimal guidance
Naive	Trigonometrically traversing the domain
Null	Staying at initial location

Simulation: single sensor

Fixed sensor noise R=0.5



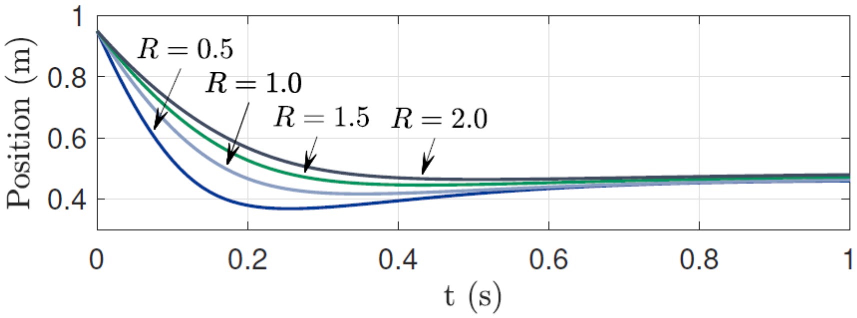


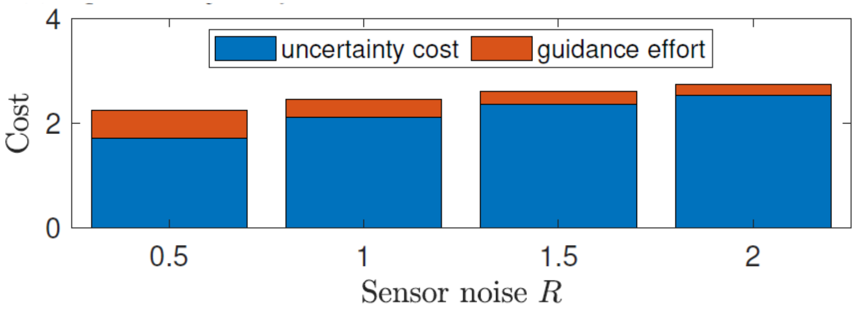


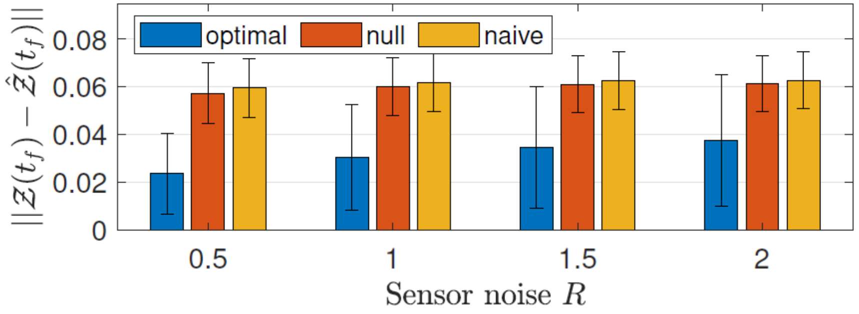
Optimal trajectory

Optimal cost

Terminal estimation error







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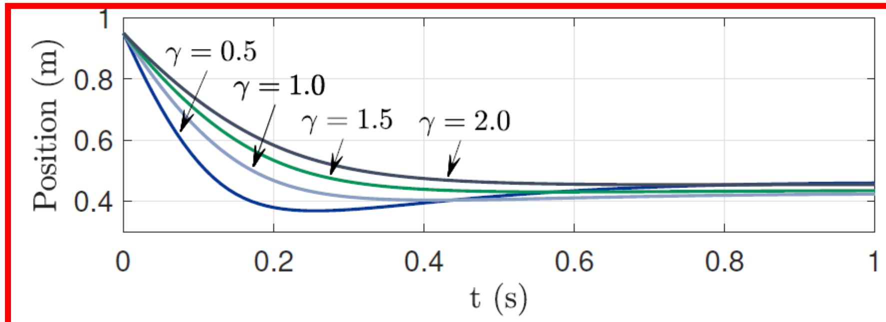

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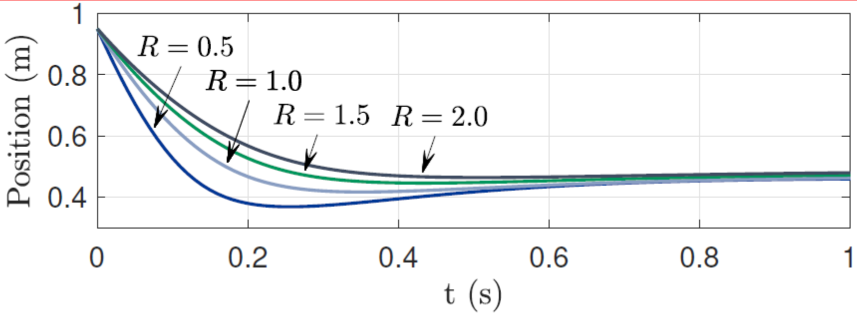

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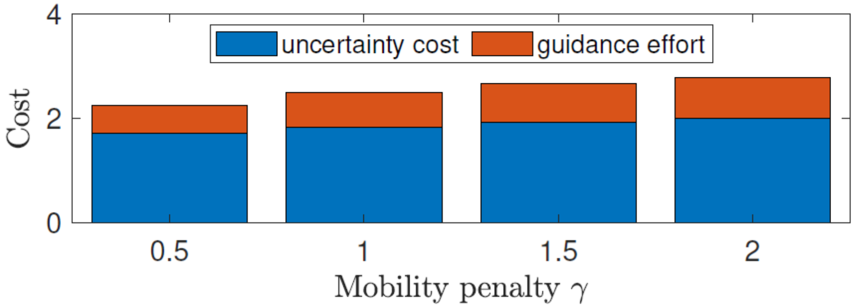

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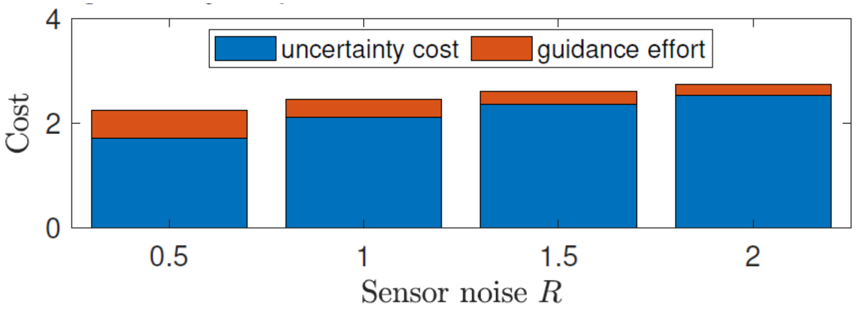

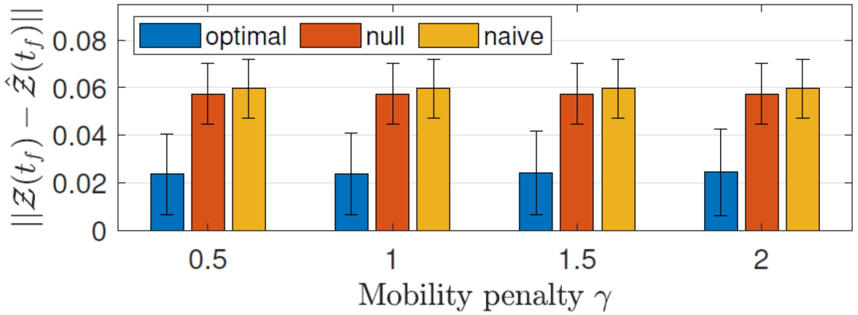

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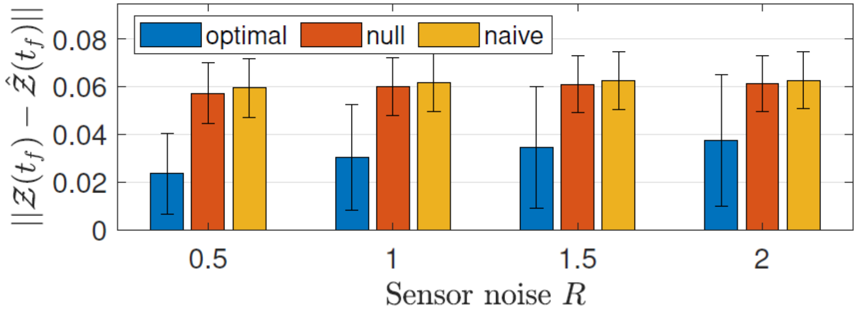

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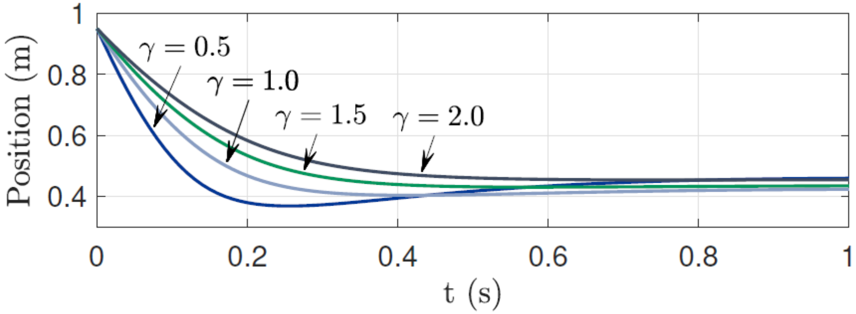

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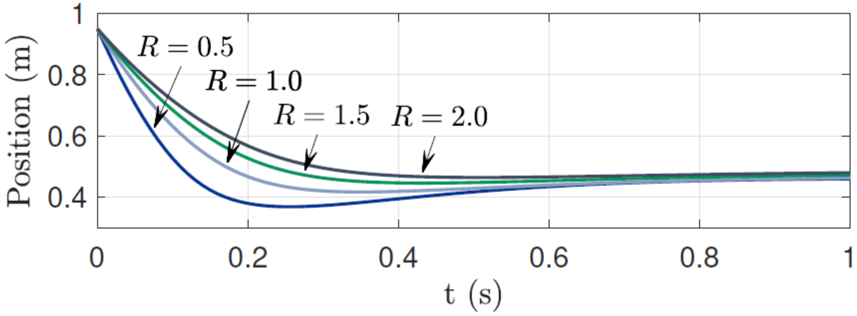

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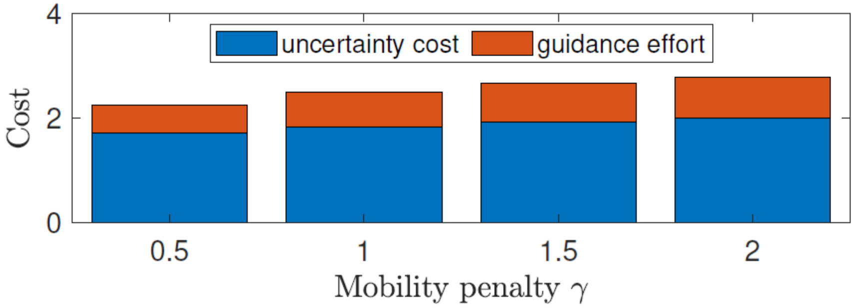

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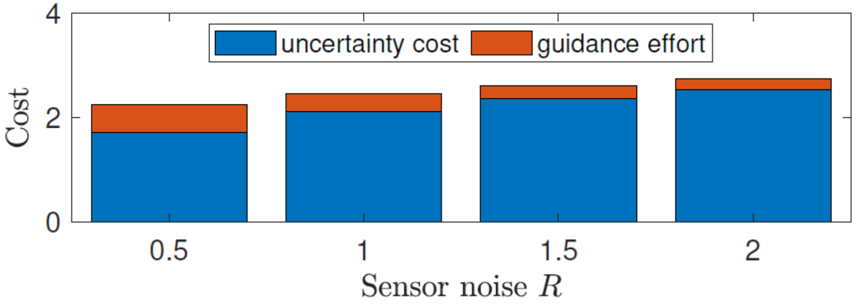

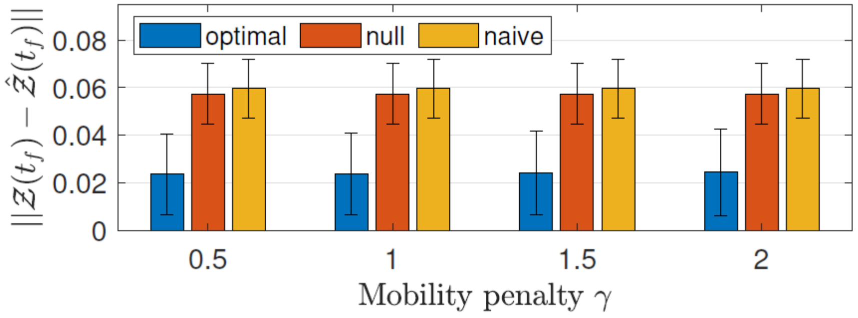

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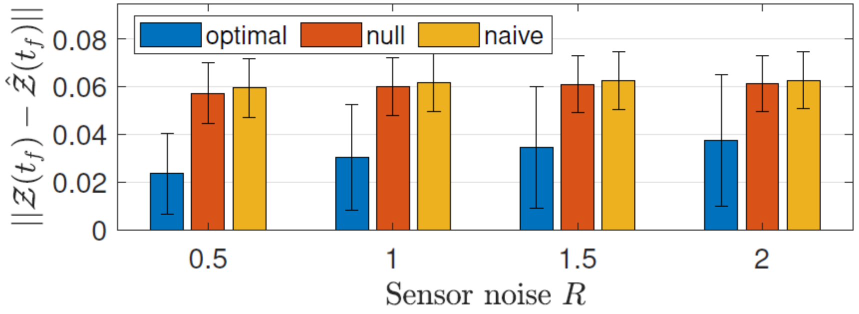

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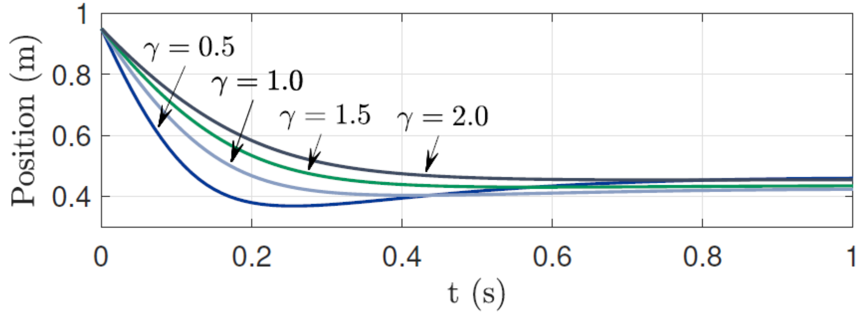

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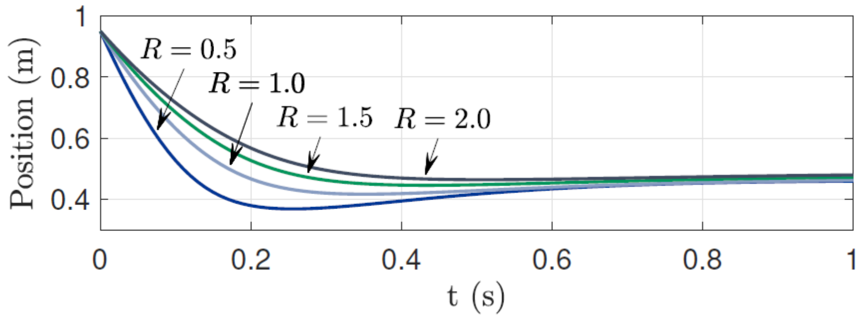

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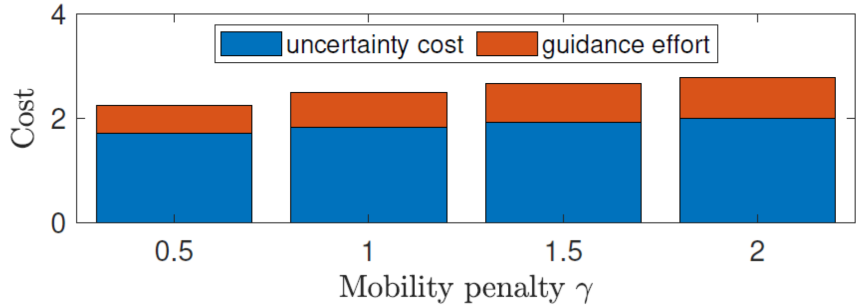

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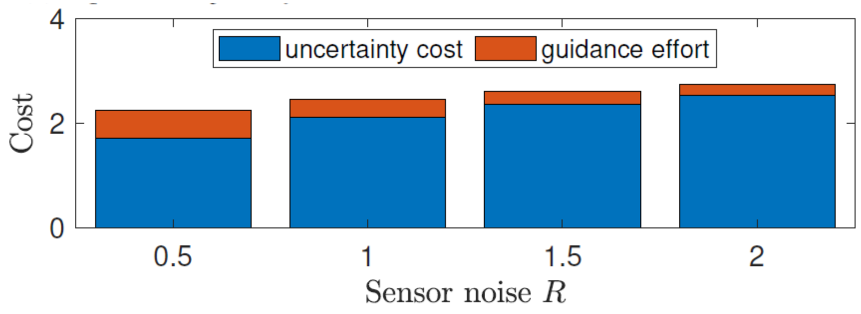

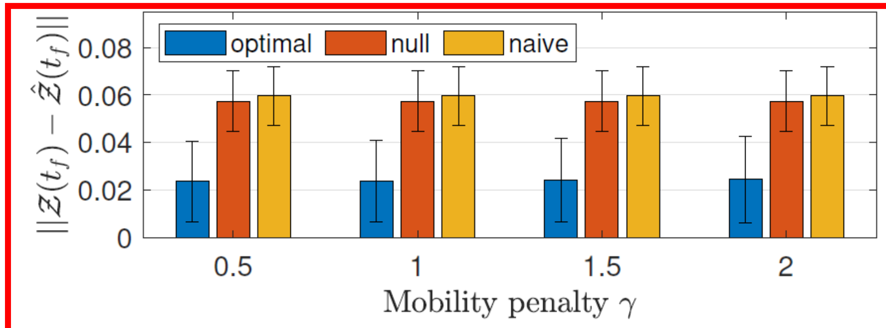

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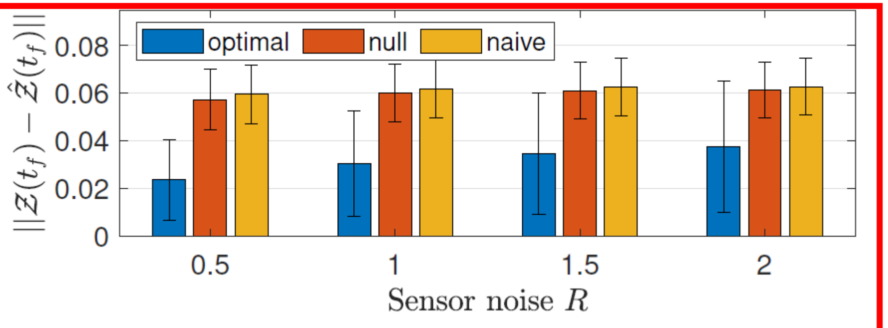

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Maximum principle

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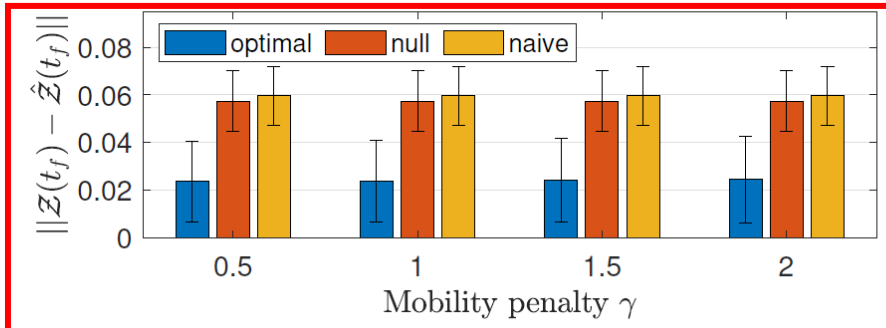
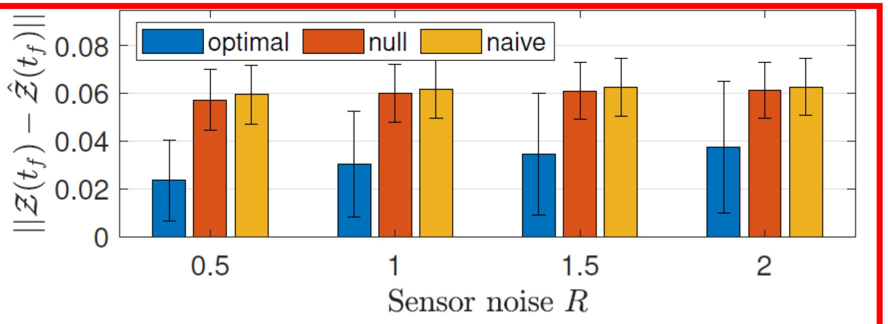
Kalman filter



Terminal estimation error



Simulation: single sensor

Fixed sensor noise R=0.5

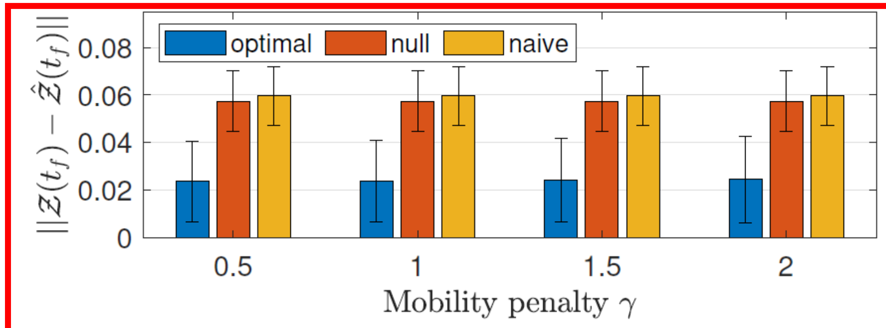
Fixed mobility penalty γ=0.5

$$\dot{\zeta}^*(t) = a\zeta^*(t) + bp^*(t),\ \ \zeta^*(0) = \zeta_0,$$
$$\dot{\lambda}^*(t) = -a^T\lambda^*(t) - (\nabla_{\zeta^*}\mathrm{Tr}(\mathcal{P}(t)))^T,\ \ \lambda^*(t_f) = 0,$$
$$p^*(t) = -\gamma^{-1}b^T\lambda^*,$$

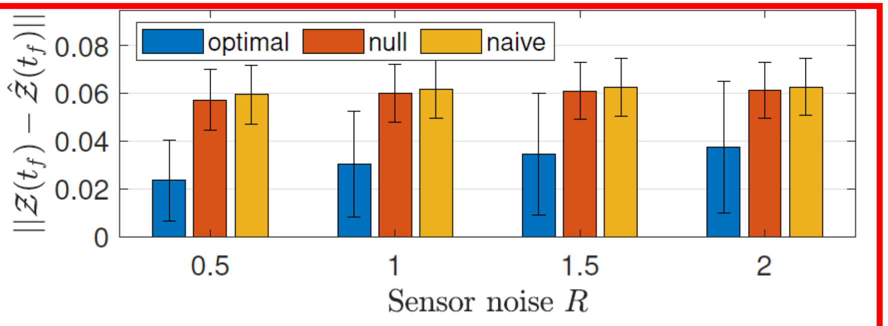
Maximum principle

$$\dot{\hat{\mathcal{Z}}}(t) = \mathcal{A}\hat{\mathcal{Z}}(t) + \mathcal{P}(t)\mathcal{C}_{\zeta(t)}\boxed{R^{-1}}(y(t) - \hat{y}(t))$$
$$\hat{\mathcal{Z}}(t_0) = \hat{\mathcal{Z}}_0,$$

Kalman filter



Terminal estimation error



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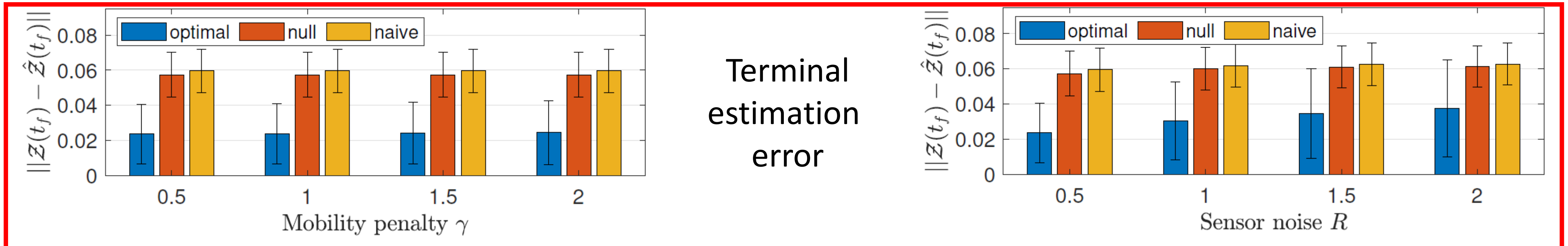
Maximum principle

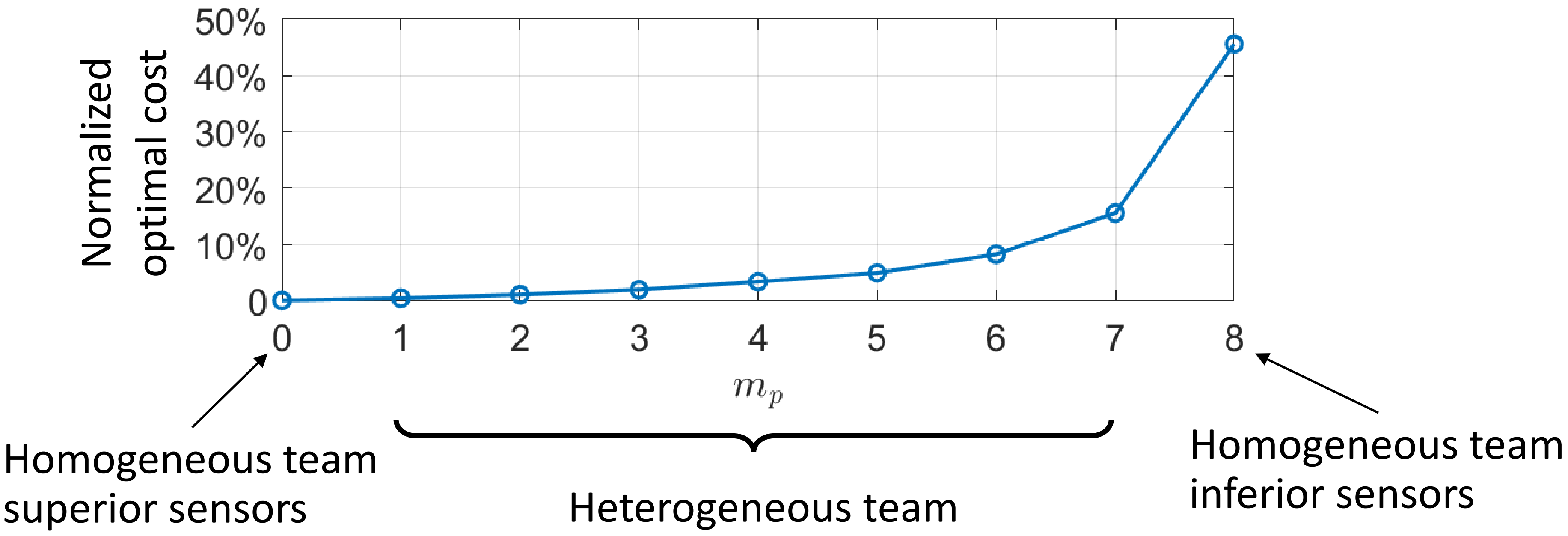
$$\dot{\hat{\mathcal{Z}}}(t) = \mathcal{A}\hat{\mathcal{Z}}(t) + \mathcal{P}(t)\mathcal{C}_{\zeta(t)}R^{-1}(y(t) - \hat{y}(t))$$
$$\hat{\mathcal{Z}}(t_0) = \hat{\mathcal{Z}}_0,$$

Kalman filter



Simulation: Heterogeneous team

- Assume a larger investment is required for a superior sensor ($R = 0.5$ and $\gamma = 0.5$) than for an inferior sensor ($R = 2$ and $\gamma = 1$).
- Heterogeneous team: m_p inferior sensors + $(8 - m_p)$ superior sensors.



Summary and ongoing work

- Estimation of a diffusion process using a team of mobile sensors under optimal guidance

- Ongoing work
  - Extend the framework to a diffusion-advection process with 2D spatial domain
  - More efficient numerical computation
  - Convergence of the approximate optimal solution*.
  - Simultaneous estimation and control with a team of mobile sensor-plus-actuators.

*S. Cheng and D. A. Paley, “Optimal control of a 2D diffusion-advection process with a team of mobile actuators under jointly optimal guidance,” submitted.

Thank you!

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