Reaching a target in a time-costly area using a two-stage optimal control method

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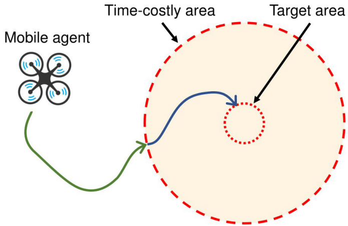
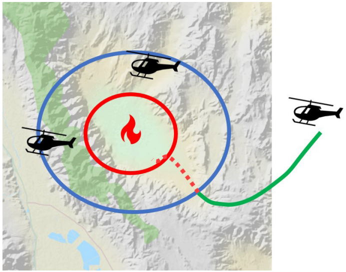
Target reaching inside a time-costly area

- Problem of study: optimally reaching a target enclosed within a time-costly area
- Motivating scenario: helicopter rescue
- Proposed solution: two-stage optimal controller
  - Outer stage
  - Inner stage
- Related work: necessary condition of optimality with an adjustable intermediate time (Tomiyama, JEDC, 1985)





Contributions

- Formulate the problem and transform the infinite-dimensional problem to a finite-dimensional one
- Prove that the transformed problem can be efficiently solved using a solution of its relaxation under certain symmetry conditions
    - Optimality gap test (Cheng and Martins, arXiv preprint, 2019)

Outline

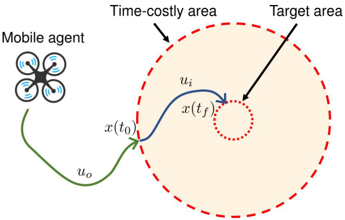
Outline

Problem setup

- Time-costly area and target area are 'elliptical'.
- Mobile agent: linear dynamics

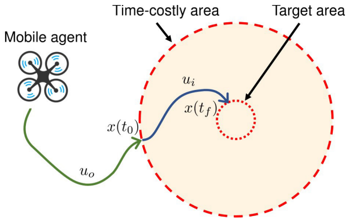
$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0.$$

- Outer stage control u_o and inner stage control u_i
- Switching time t_0 and switching state $x(t_0)$
- Terminal time t_f and terminal state $x(t_f)$
- Known: (A, B) reachable, x_0, and t_0
- Variable: t_f, $u(\cdot)$ and $x(\cdot)$



Formulation

$$\begin{aligned}
\underset{u\in\mathcal{U}(0,t_f),t_f\in\mathbb{R}}{\text{minimize}} \quad & \frac{1}{2}\int_0^{t_0} \|u_o(t)\|_R^2 + \|x(t)\|_Q^2 \,\mathrm{d}t \\
& + \frac{1}{2}\int_{t_0}^{t_f} \|u_i(t)\|_R^2 \,\mathrm{d}t + \phi(t_f - t_0) \\
\text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\
& u(t) = \begin{cases} u_o(t), \text{ if } t \in [0, t_0), \\ u_i(t), \text{ if } t \in [t_0, t_f), \end{cases} \\
& \|x(t_0)\|_D^2 = d_1^2, \\
& \|x(t_f)\|_D^2 \leq d_2^2, \\
& t_f \in (t_0, t_0 + T].
\end{aligned}$$



$\phi(\cdot)$: continuous and increasing

Subproblems

- Inner stage problem $\longrightarrow u_i^*$

$$
\begin{aligned}
J_i^*(t_f, x(t_0)) \overset{\text{def}}{=} \min_{u_i \in \mathcal{U}(t_0, t_f)} \quad & \frac{1}{2}\int_{t_0}^{t_f} \|u_i(t)\|_R^2 \,\mathrm{d}t \\
\text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu_i(t), x(t_0) \text{ given}, \\
& \|x(t_f)\|_D^2 \le d_2^2.
\end{aligned}
$$

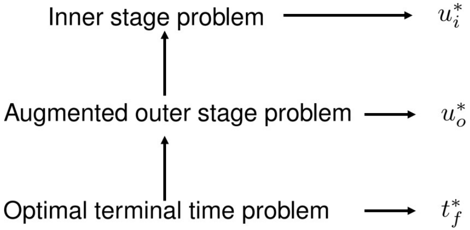
- Augmented outer stage problem $\longrightarrow u_o^*$

$$
\begin{aligned}
J^*(t_f) \overset{\text{def}}{=} \min_{u_o \in \mathcal{U}(0, t_0)} \quad & \frac{1}{2}\int_{0}^{t_0} \|u_o(t)\|_R^2 + \|x(t)\|_Q^2 \,\mathrm{d}t + J_i^*(t_f, x(t_0)) \\
\text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu_o(t), x(0) = x_0, \\
& \|x(t_0)\|_D^2 = d_1^2.
\end{aligned}
$$

- Optimal terminal time problem $\longrightarrow t_f^*$

$$
\underset{t_f \in (t_0, t_0 + T]}{\text{minimize}} \quad J^*(t_f) + \phi(t_f - t_0).
$$

Flowchart


Inner stage problem
u_i^*
Augmented outer stage problem
u_o^*
Optimal terminal time problem
t_f^*

Outline

Inner stage problem

- **Minimum control effort** subject to **linear dynamics** and a **terminal state constraint**

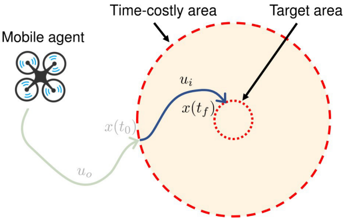
$$\begin{aligned} \underset{u_i \in \mathcal{U}(t_0, t_f)}{\text{minimize}} \quad & \frac{1}{2}\int_{t_0}^{t_f} \|u_i(t)\|_R^2 \, \mathrm{d}t \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu_i(t), x(t_0) \text{ given}, \\ & \|x(t_f)\|_D^2 \le d_2^2. \end{aligned}$$

- Equivalent problem[1]: find an optimal terminal state $r^*(t_f)$

$$\begin{aligned} \underset{r(t_f) \in \mathbb{R}^4}{\text{minimize}} \quad & \frac{1}{2}\|x_f - r(t_f)\|_{\Delta^{-1}(t_f)}^2 \\ \text{subject to} \quad & \|r(t_f)\|_D^2 \le d_2^2. \end{aligned}$$

- Optimal (open-loop) control:

$$u_i^*(t) = -R^{-1}B^T e^{A^T(t_f - t)}\Delta^{-1}(t_f)(x_f - r^*(t_f)), t \in [t_0, t_f).$$

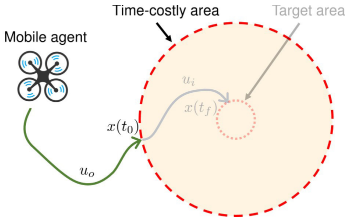


[1] Lewis, Vrabie, and Syrmos, Optimal control, 2012

Augmented outer stage problem

- Augmented outer stage problem:

$$\begin{aligned} \underset{u_o \in \mathcal{U}(0,t_0)}{\text{minimize}} \quad & \frac{1}{2}\int_0^{t_0} \|u_o(t)\|_R^2 + \|x(t)\|_Q^2 \,\mathrm{d}t + J_i^*(t_f, x(t_0)) \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + Bu_o(t), x(0) = x_0, \\ & \|x(t_0)\|_D^2 = d_1^2. \end{aligned}$$



- LQR theory[1]: the optimal control is a **linear** combination of the **state** and **switching state**.
- Equivalent problem: find an optimal switching state $r^*(t_0)$ and an optimal terminal state $r^*(t_f)$.
- Optimal control: $u_o^*(t) = -H(t)x^*(t) - L(t)\, r^*(t_0)$.

[1] Lewis, Vrabie, and Syrmos, Optimal control, 2012

Augmented outer stage problem

- Equivalent problem: find an optimal switching state $r^*(t_0)$ and an optimal terminal state $r^*(t_f)$

$$\begin{aligned}\underset{r(t_0)\in\mathbb{R}^4, r(t_f)\in\mathbb{R}^4}{\text{minimize}} \quad & \| r(t_0) \|^2_{\Xi_3} + 2x_0^T \Xi_2 r(t_0) + \|x_0\|^2_{\Xi_1} + \frac{1}{2}\|\Phi(t_f, t_0) r(t_0) - r(t_f) \|^2_{\Delta^{-1}(t_f)} \\ \text{subject to} \quad & \| r(t_0) \|^2_D = d_1^2, \\ & \| r(t_f) \|^2_D \leq d_2^2.\end{aligned}$$

- Rewrite the problem: concatenate optimization variables $y^T = \begin{bmatrix} r^T(t_0) & r^T(t_f) \end{bmatrix}$.

$$\begin{aligned}\underset{y\in\mathbb{R}^8}{\text{minimize}} \quad & \|y\|^2_M + 2q^T y \\ \text{subject to} \quad & \|y\|^2_{\Gamma_1} = d_1^2, \\ & \|y\|^2_{\Gamma_2} \leq d_2^2.\end{aligned}$$

Quadratic program with two quadratic constraints (QC2QP)!

Solution method: relaxations of QC2QP

$M \succ 0$, relax the equality constraint:

$$\begin{array}{ll} \underset{y\in\mathbb{R}^8}{\text{minimize}} & \|y\|_M^2 + 2q^T y \\ \text{subject to} & \|y\|_{\Gamma_1}^2 = d_1^2, \\ & \|y\|_{\Gamma_2}^2 \leq d_2^2. \end{array}$$

$$\begin{array}{ll} \underset{y\in\mathbb{R}^8}{\text{minimize}} & \|y\|_M^2 + 2q^T y \\ \text{subject to} & \|y\|_{\Gamma_1}^2 \leq d_1^2, \\ & \|y\|_{\Gamma_2}^2 \leq d_2^2. \end{array}$$

Convex!

$$\begin{array}{ll} \underset{y\in\mathbb{R}^8}{\text{minimize}} & \|y\|_M^2 + 2q^T y \\ \text{subject to} & \|y\|_{\Gamma_1}^2 \geq d_1^2, \\ & \|y\|_{\Gamma_2}^2 \leq d_2^2. \end{array}$$

Nonconvex: use optimality gap test in QC2QP (Cheng and Martins, ArXiv preprint, 2019.)

QC2QP: optimality gap test

Theorem 1 (Theorem 3.2, Cheng and Martins, arXiv preprint, 2019)

Globally optimal solution of QC2QP using a semidefinite relaxation: check four algebraic conditions of solutions of the semidefinite relaxation and the Lagrange dual

- Goal: QC2QP solved by semidefinite relaxation (Y/N)
- Do: solve two convex problems
- Check: four algebraic conditions
- $>60\%$ nonconvex QC2QPs have no gap

Theorem 2

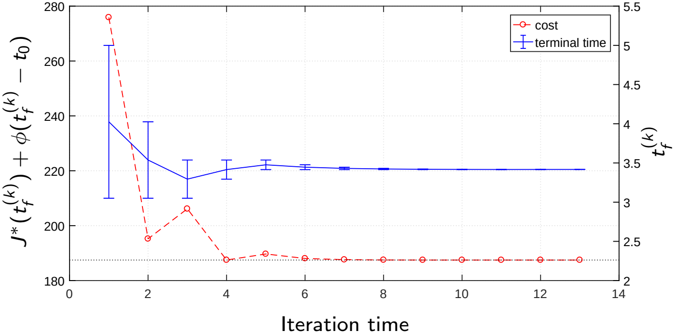
The dynamics, cost evaluation, and time-costly/target area are **symmetric** *and* **decoupled** *along the horizontal axis and the vertical axis in the 2D plane: globally optimal solution of QC2QP guaranteed from its semidefinite relaxation.*

Optimal terminal time

- First-order necessary condition: search for a locally optimal terminal time t_f^*:

$$\dot{J}^*(t_f^*) + \dot{\phi}(t_f^* - t_0) = 0.$$

- Bisection: $\dot{J}^*(t_f)$ can be evaluated only when optimization is solved with a specific t_f.

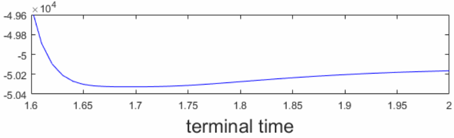


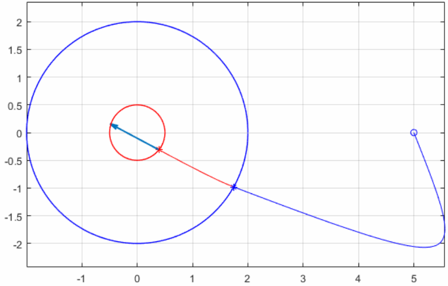
Outline

Optimal trajectory v.s. sign of $\dot{J}^*(t_f)$

- Assumption: the control u can only affect the acceleration.
- Optimal trajectory at terminal time v.s. the sign of $\dot{J}^*(t_f)$.

Sign	Behavior
$\dot{J}^*(t_f) < 0$	Entering
$\dot{J}^*(t_f) > 0$	Exiting
$\dot{J}^*(t_f) = 0$	Tangential





Outline

Conclusion

- Two-stage optimal control problem: find an optimal switching state and an optimal terminal state
- Equivalent problem as QC2QP: symmetry guarantees exact solution from a semidefinite relaxation
- Property: optimal trajectory v.s. derivative of optimal cost $J^*(t_f)$

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- Sheng Cheng: cheng@terpmail.umd.edu

- S. Cheng and N. C. Martins, "An Optimality Gap Test for a Semidefinite Relaxation of a Quadratic Program with Two Quadratic Constraints," arXiv:1907.02989, 2019.